Essential Calculus

with Motivational Problems

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Preface

This book is titled Essential Calculus for a reason. It is written to present the basics of calculus in an easy to understand way. If you want to be exposed to an overview of calculus, with enough depth to provide understanding and the ability to solve real world problems, I hope that this book will satisfy that urge.

Calculus is a subject which can change the way that one looks at the world. It is not difficult and can be eye-opening and exciting to learn. There are many "AHA!" moments available to the careful student of calculus.

I hope that this book will allow me to share some of those moments with you.

Why did I write this book? My son is 22 and for various reasons still has not taken a calculus course. My background is in Electrical Engineering and I have anticipated teaching my son calculus ever since the two of us would jog together doing multiplication tables when he was 7.

My son has taken a calculus pre-requisite class in college, but is focused on other areas and still has not taken the course. Perhaps he will find this book useful when he finally takes Calculus.

Bob Smither
February, 2010

2. Motivation

This chapter presents several motivational problems that can be solved using the ideas developed in this book. It is hoped that by presenting these problems early in the book you will be motivated to continue so you can learn and appreciate how real world problems can be solved using calculus. The solutions to each of the motivational problems will be presented later in the book after the material required for their solution has been covered.

Don't be surprised if you do not understand all of the ideas presented in these problems! You are not expected to. Complete explanations and solutions will be presented after we have covered the calculus ideas needed to solve each Motivational Problem.

2.1. Slope of a Curve

We are all familiar with the slope - intercept representation of a straight line [3]:

$$y = m * x + b$$

where m is the slope of the line and b is the y axis intercept (the value of y when x = 0). The slope, m, is the ratio of

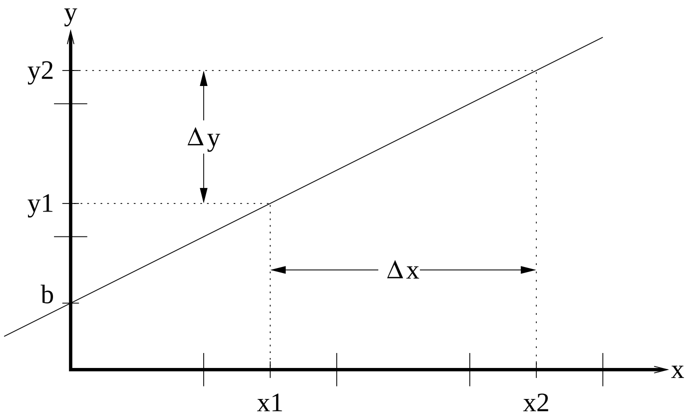

Figure 2.1.: Straight Line Plot

the *change* in the y direction divided by the *change* in the x direction as one moves along the line. In other words:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

where the Δ is measured over some convenient length. See Figure 2.1 for the details.

Measuring the slope of a straight line is pretty straightforward, but what about the slope of a *curve*? Looking at Figure 2.2 it is clear that the slope of the line changes as x changes - in fact the slope can be seen to be a function of x.

Motivational Problem 2.1 is to find the function of x that is the slope of the curve shown in Figure 2.2 for any value of x.

Solution: Section 3.2 on page 37.

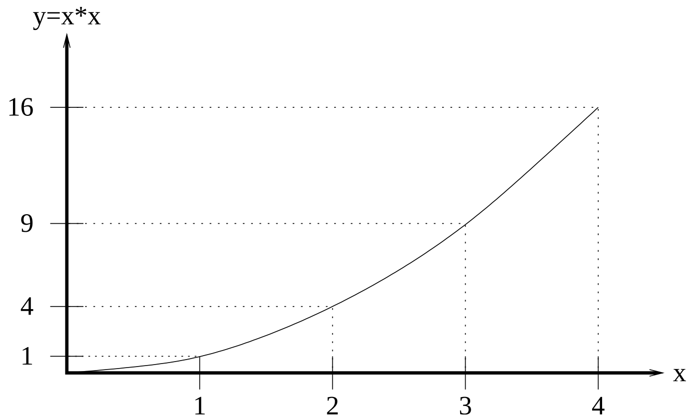

Figure 2.2.: X Squared Curve

2.2. The Capacitor

A capacitor is a two terminal electrical element that is used to store electrical charge. Capacitors are used in many industrial applications and in just about all electronic devices. The basic operation of a capacitor is described by

$$Q = C * V$$

where Q is the charge stored on the capacitor, V is the voltage measured across the capacitor's terminals, and C is a measure of the "capacitance" of the capacitor. The units are: Coulombs for Q, Volts for V,and Farads for C.

The time rate of charge movement in a conductor is current. Denoting current as I and noting that current in the direction defined in Figure 2.3 will decrease the charge we have

$$I = -dQ/dt.$$

Current is measured in Amperes. The flow of charge, that is current, through a resistance (R) causes a voltage difference

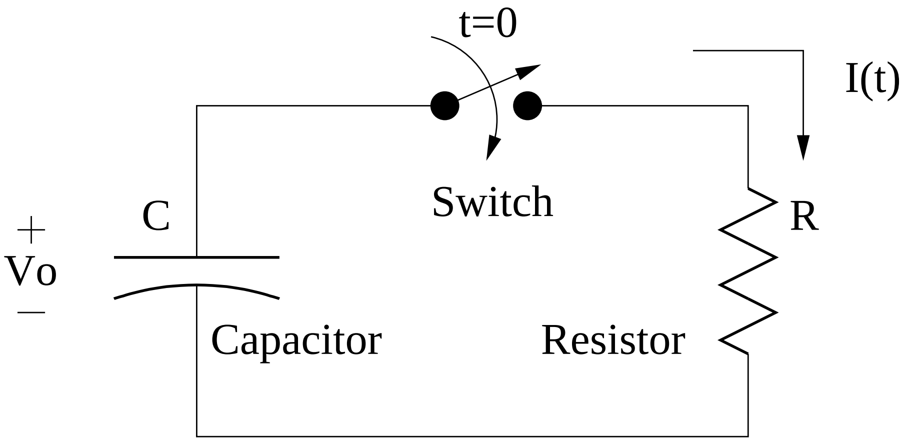

Figure 2.3.: Discharging a Capacitor

to appear between the ends of the resistor. The equation expressing this fact is known as Ohm's Law:

$$V = I * R$$

where V is the voltage drop between the ends of the resistor. Resistance is measured in Ohms.

Figure 2.3 illustrates an experiment in which a capacitor of capacitance C, originally charged to V_o Volts, is connected across a resistor of resistance R at time $t = 0$ by closing the switch. Before time $t = 0$ the switch is open, the voltage on the capacitor is V_o, and the current $I(t) = 0$.

Motivational Problem 2.2 is to find the equation for the current $I(t)$ that flows through the resistor R for time ≥ 0.

Solution: Section 3.8 on page 49.

2.3. Age of the Earth[5]

The atoms of certain elements spontaneously decay through emission of radiation or particles. While each decay event is random, given a large number of radioactive atoms the average rate is predictable. Clearly the more such atoms there are the more will decay, so we expect the rate at which radioactive atoms are lost to be proportional the number of such atoms. Remember that the derivative relates to a rate, so it is natural to write

$$dN/dt = -\lambda * N. \tag{2.1}$$

(see Section 3.1 for an introduction to the notation used here) The constant of proportionality, λ, is positive so the rate, dN/dt, is a negative number implying that the number of radioactive atoms decreases as time progresses.

It turns out that two isotopes of uranium, ^{235}U and ^{238}U, decay at different rates. It also develops, as best we know, that the two isotopes should have been created in approximately equal quantities and their abundance should have been approximately equal around the time the earth was forming. The current ratio of ^{238}U to ^{235}U on earth is 137.8. The λ for ^{235}U is 9.80E-10 and the λ for ^{238}U is 1.55E-10.

Motivational Problem 2.3 is to estimate the age of the earth from the above information.

Solution: Section 3.9 on page 51.

2.4. Designing Pastures

Pastures are often defined by fences, and in designing a pasture for livestock an important consideration is the cost.

Let's assume that fencing costs a fixed amount per foot, say $\$F/ft.$, and that a rancher has a budgeted amount, say $\$B$, that he is prepared to spend on fencing a new, rectangular, pasture. Motivational Problem 2.4 is: How should the pasture be designed so that the rancher gets the largest possible fenced area for his expenditure of $\$B$?

Solution: Section 3.12 on page 56.

2.5. Maximizing Profit

Let's assume we want to manufacture and sell some Widgets (W). The market for Widgets has a demand curve as shown in Figure 2.4 - that is, we can only sell more Widgets if we are willing to lower the price. The Figure tells us that we will sell no Widgets at \$10.00/W, but can increase our sales by 100W each time we lower the price by \$2.50. The demand curve has a slope of -\$0.025/W.

There is also a cost curve associated with producing and selling Widgets as shown in Figure 2.5. After a fixed cost of \$500, each Widget costs us \$2.50 to manufacture.

Motivational Problem 2.5 is to determine the number of Widgets we should make in order to maximize our profit.

Solution: Section 3.14 on page 58.

2.6. Euler's Formula

Motivational Problem 2.6 is to demonstrate the truth of Euler's Formula, which is:

$$\exp(j * \theta) = \cos(\theta) + j * \sin(\theta).$$

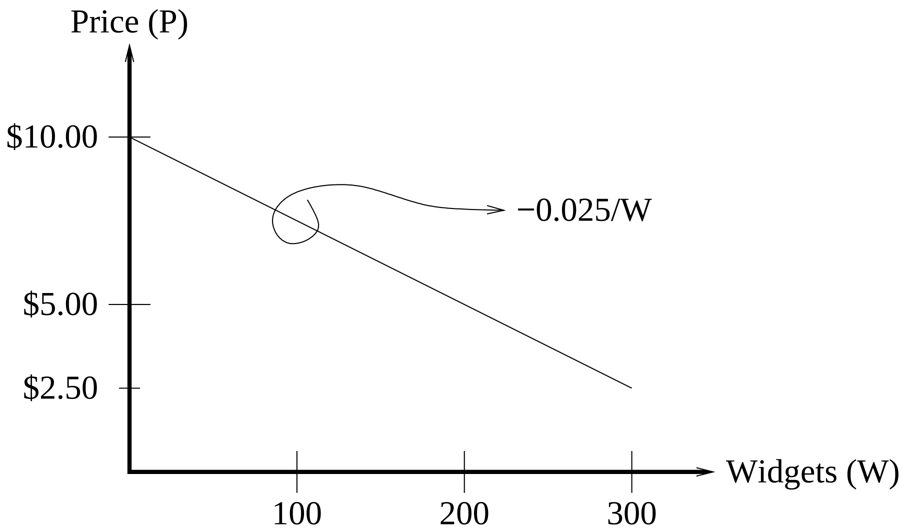

Figure 2.4.: Price versus Widgets - Demand Curve

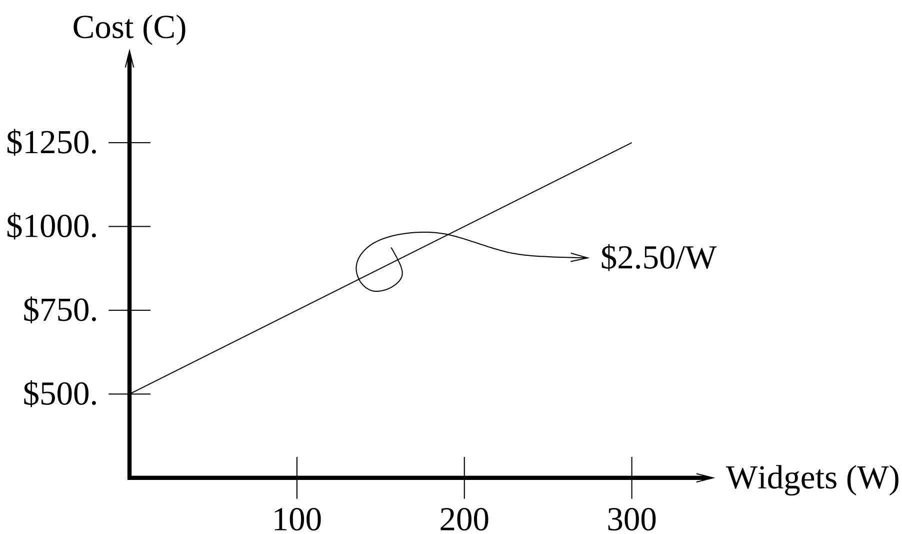

Figure 2.5.: Cost versus Widgets - Cost Curve

Solution: Section 3.16 on page 62

2.7. Maximum Package Volume

In the United States the UPS company restricts packages to a combined girth plus length of 165 inches with a maximum allowed length of 108 inches. Length is defined as the length of the longest side of the rectangular package. If we want to ship a large quantity of small items, we want to maximize the volume of the package that we use for shipments.

Motivational Problem 2.7 is to find the dimensions of the largest volume package that UPS will ship for us.

Solution: Section 3.19 on page 65.

2.8. What is the value of $0/0$?

We sometimes run into expressions such as

$$f(x) = 3 * \sin(x)/x$$

and we need to know the value of $f(x)$ when x goes to 0.0. In other words we need to find

$$lim \mid_{x \to 0} 3 * \frac{\sin(x)}{x}.$$

Substituting the limit value of 0 gives $\frac{0}{0}$ - not very useful. Motivational Problem 2.8 is to find the value of the above limit.

Solution: Section 3.23 on page 73.

2.9. What is the value of 1^∞?

In Section 1.4 we developed this limit:

$$\lim |_{h \to 0} (1 + h)^{1/h}$$

which we declared was equal to the transcendental number e. Substituting the limit value of 0 results in 1^∞ - not easily understood. Motivational Problem 2.9 is to show that indeed this limit equals e.

Solution: Section 3.25 on page 75.

2.10. Acceleration, Speed, Distance

If an object is accelerated at a constant acceleration a (with units of $m/sec/sec$) it's speed will increase by $a(m/sec)$ for each second that it is accelerated. For example, an object dropped from a tower on earth experiences an acceleration of $9.8m/sec/sec$ so that three seconds after being dropped it will be moving at $29.4m/sec$ (ignoring the effect of air drag and assuming that the object has not hit the ground!).

Motivational Problem 2.10 is to obtain the equation for the speed and the distance traveled by an object subject to a constant acceleration a.

Solution: Section 4.3 on page 87.

2.11. The Longest Shot

In firing a cannon one of the variables that can be controlled is the angle that the cannon makes with the horizontal. Fig-

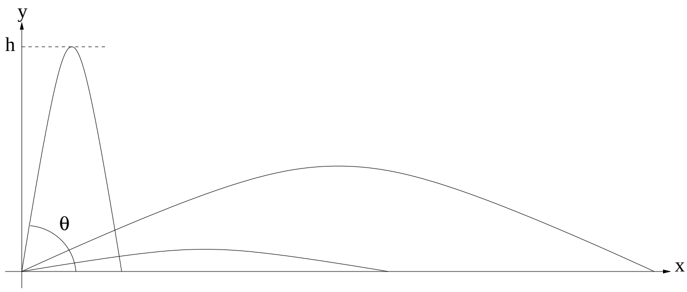

Figure 2.6.: The Longest Shot

The Longest Shot

ure 4.5 illustrates the situation being considered - a cannon is fired on a flat plane. Each time the cannon is fired, the cannon ball is the same size and weight and the initial speed of the ball is the same. Motivational Problem 2.11 is, ignoring wind resistance and assuming that gravity is constant, find the angle Θ that results in the longest possible shot from the cannon.

Solution: Section 4.4 on page 91.

2.12. Distance Traveled

Motivational Problem 2.12 is to find the distance traveled after 60 seconds if the speed versus time curve is as shown in Figure 2.7.

Solution: Section 4.5 on page 97

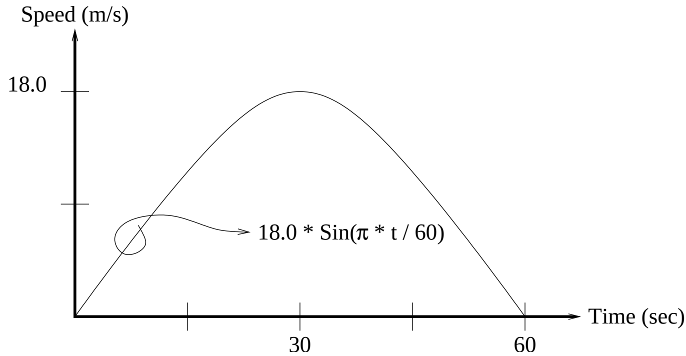

Figure 2.7.: Variable Speed vs. Time

2.13. Area of a Circle

The area of a square is easy, as we saw in Figure 4.8, but how about the area of a circle? Figure 2.8 shows a circle of radius R. We know from earlier math classes that the area of the circle is $\pi * R^2$, but how can this be demonstrated?

Motivational Problem 2.13 is to show that the area of a circle with a radius of R is

$$A = \pi * R^2.$$

The Figure gives a hint of how we will go about solving this problem.

Solution: Section 4.6 on page 103.

2.14. Potential Energy of Gravity

Gravity near the earth's surface exerts a constant acceleration on mass. The force of gravity, $F = m * g$ (Force (F)

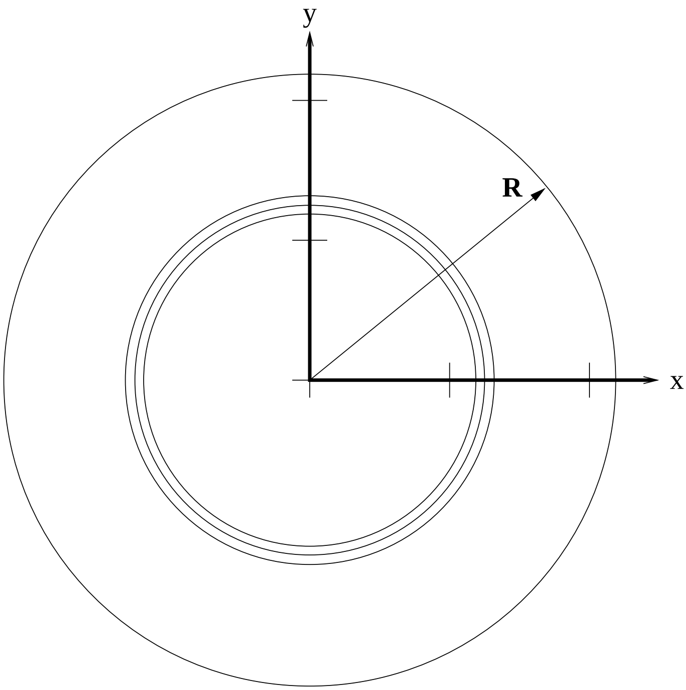

Figure 2.8.: Circle of Radius r

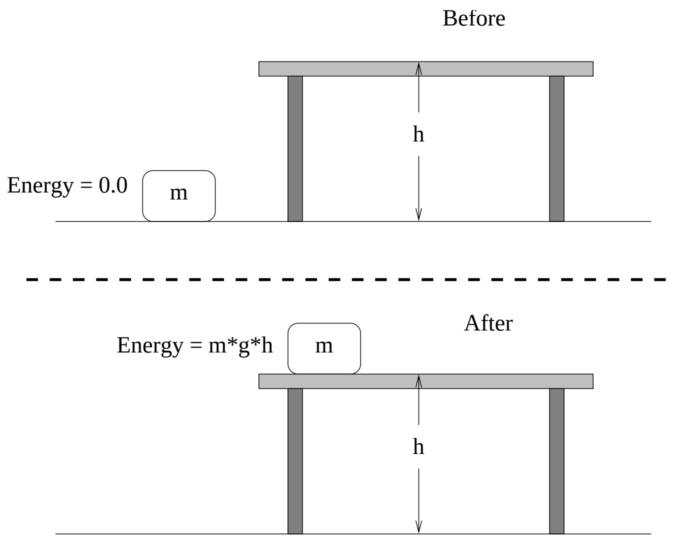

Figure 2.9.: Gravitational Potential Energy

equals mass (m) times acceleration (g)), is what we experience as our weight. To a physicist, work is equal to force times the distance that the force acts over. If we lift a mass against the force of gravity we do work on the mass and give it potential energy. In the case of constant acceleration, as seen near the earth's surface, the force is constant and the potential energy is given by $m * g * h$ where g is the (constant) acceleration of gravity and h is the height that the object is lifted. See Figure 2.9.

What happens when the force of gravity is not constant? As we go away from the earth's surface, the acceleration of gravity reduces. The simple analysis above that takes the acceleration of gravity g to be constant is no longer right. Figure 2.10 illustrates the new situation.

Motivational Problem 2.14 is to find an expression for the work done on a mass m as it is removed from the vicinity of

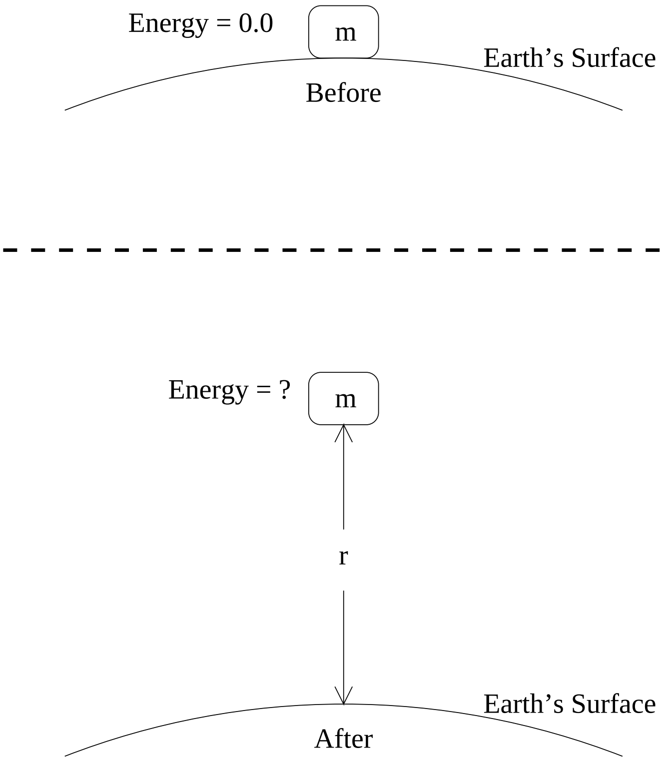

Figure 2.10.: Gravitational Potential Energy

the earth, or what amounts to the same thing, find the potential energy due to gravity for a mass at an arbitrary distance h measured from the earth's surface. Energy is always defined relative to an arbitrary reference, so you can assume that the energy of the mass resting on the earth's surface is zero. Use the resulting expression to find the speed needed by a mass m at the earth's surface to completely escape the pull of earth's gravity (complicating issues like the drag of the earth's atmosphere can be ignored!).

Solution: Section 4.8 on page 107.

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